V. *On the Theory of Free Stream Lines.*

By J. H. MICHELL, *Trinity College, Cambridge.*

Communicated by Professor J. J. THOMSON, *F.R.S.*

Received January 3,—Read January 16, 1890.

Introduction.

THE attention of mathematicians was first called to the subject of the present paper by a memoir of HELMHOLTZ'S in 1868, on "Discontinuous Fluid Motion."*

In discussing the steady motion of liquids past salient edges of fixed obstacles, it is found that the assumption of continuity of the motion leads to negative pressures in the liquid. HELMHOLTZ showed, in the paper above-mentioned, that some cases of this kind could be solved by assuming a surface of discontinuity, on one side of which the liquid is at rest, and he gave a mathematical solution of one case where the motion is in two dimensions.

The next advance in the subject was made by KIRCHHOFF who, in 1869, in a paper entitled "Zur Theorie freier Flüssigkeitsstrahlen" in 'CRELLE'S Journal,' gave a generalization of the method which HELMHOLTZ had used, and obtained thereby the solution of three new interesting cases. Subsequently in his 'Vorlesungen über mathematische Physik,' he published another method and worked out the same problems by means of it, but gave no new ones.

RAYLEIGH in the 'Philosophical Magazine,' December, 1876, discussed the solutions of KIRCHHOFF, and gave a drawing of the bounding free stream lines in one case.

As far as I know, these are the only investigations published on the mathematical side respecting a branch of hydrodynamics of great theoretical and practical interest.

In considering the method of transformation of polygons given independently by SCHWARZ and CHRISTOFFEL I have been led to a new transformation, which together with theirs, gives a general solution of the problem of free non-reentrant stream lines with plane rigid boundaries.

A considerable number of the cases of high interest prove to be of a tolerably simple nature, and I have worked out several in detail.

These problems occupy the first part of the paper. In the second part I have given some extensions of the transformation formulæ, which are applicable to problems of condensers and the form of hollow vortices in certain cases.

* 'Berlin Monatsberichte,' 1868; and 'Gesamm. Abhandl.,' vol. 1.

28.6.90

The general Theory of Transformation.

Let x, y be two conjugate functions with respect to the two variables ϕ, ψ, so that

$$x + iy = f(\phi + i\psi),$$

and write

$$x + iy = z \qquad x - iy = z',$$

$$\phi + i\psi = w \qquad \phi - i\psi = w'.$$

x, y may be regarded as the rectangular coordinate of a point in a plane which we shall call the z plane, and similarly ϕ, ψ are the coordinates of a point in the w plane.

Consider the functions

$$\mathrm{V} = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

$$\mathrm{W} = -i \log \frac{dz}{dw} \Big/ \frac{dz'}{dw'}.$$

Since they can be written in the form

$$\mathrm{V} = \log \frac{dz}{dw} + \log \frac{dz'}{dw'},$$

$$\mathrm{W} = -i\left[\log \frac{dz}{dw} - \log \frac{dz'}{dw'}\right],$$

they both satisfy LAPLACE's equation, and we have

$$\mathrm{V} + i\mathrm{W} = 2 \log \frac{dz}{dw},$$

so that V, W are conjugate functions with respect to x, y or ϕ, ψ.

The transformations of the present paper will be deduced from the properties of the function V, so that its nature must be considered in detail. We have as alternative forms of V

$$\mathrm{V} = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'} = \log \left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\} = -\log \left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\}.$$

If the element of arc of ψ constant in the z plane be given by

$$ds_1 = \frac{d\phi}{h},$$

then

$$h^2 = \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2,$$

so that

$$\mathrm{V} = -\log h^2.$$

And

$$i\mathrm{W} = \log \frac{\frac{dx}{d\phi} + i\frac{dy}{d\phi}}{\frac{dx}{d\phi} - i\frac{dy}{d\phi}} = 2i\theta,$$

where θ is the angle the tangent to the curve ψ makes with the axis of x.

[Mr. BRILL has used the function W as a means of transformation ('Cambridge Phil. Soc. Proc.,' vol. 6, and 'Messenger of Math.,' August, 1889), and has thus anticipated me in one of the general theorems given in the latter part of this paper as I shall notice in the proper place. I was not acquainted with his work when I developed the method here given.]

Let k be the curvature of the curve ψ at ϕ. We have the well-known formula

$$k = \frac{dh}{d\psi}$$

which is

$$k = \frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let the arc of the curve ψ be connected with k by the equation

$$s = f(k)$$

so that

$$\frac{ds}{d\phi} = \frac{d}{d\phi} f(k),$$

or from (1)

$$e^{\frac{1}{2}\mathrm{V}} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If ψ consist of parts of straight lines we have simply

$$\frac{d\mathrm{V}}{d\psi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The formula (2) suggests a general method for finding a transformation

$$z = f(\phi + i\psi),$$

such that ψ_0 is an arbitrary curve in the z plane.

If the region within ψ_0 corresponds, point for point, to the part of the ω plane lying above $\psi = \psi_0$, the problem is reduced to finding a potential function V, which is continuous throughout the space bounded by a straight line, and such that

$$e^{\frac{1}{2}\mathrm{V}} = \frac{d}{d\phi} f\left(\frac{d}{d\psi} e^{-\frac{1}{2}\mathrm{V}}\right)$$

over that straight line

We have still to discuss the question of the singular points of V.

The function V will be finite and continuous for all points except where two branches of a ψ curve (or a ϕ curve) cut.

At such a point dz/dw is either zero or infinite, and in either case V is infinite.

It will be sufficient for our purpose to consider only the simplest singularity, that is, in which we have in the neighbourhood of a point $(\phi_0\ \psi_0)$ of this nature

$$V = n \log [(\phi - \phi_0)^2 + (\psi - \psi_0)^2] + C,$$

and therefore

$$\frac{dz}{dw} = A\ (w - w_0)^n,$$

where the value of n will depend on the nature of the singularity in question, and will be seen from the particular problems to which we proceed.

PROBLEM I.

To find the transformation $z = f(\phi + i\psi)$, which makes the area for which ψ is positive in the w plane correspond point for point to the area inside a given rectilinear polygon in the z plane. (The problem of SCHWARZ* and CHRISTOFFEL†.)

Consider the conditions which the function V must satisfy in the ω plane.

(a) There are to be no singular points for ψ positive.

(b) Along $\psi = 0$ we have $dV/d\psi = 0$.

(c) At certain points of $\psi = 0$, which correspond to the angular points of the polygon in the z plane, we have V infinite.

It is plain from this specification that the function V is (to a constant) merely the potential of masses at the singular points $\phi_1, \phi_2 \ldots$ along $\psi = 0$; and, therefore,

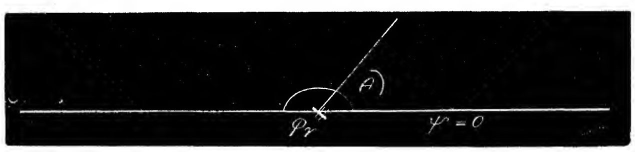
$$V = \log \Pi \{ (\phi - \phi_r)^2 + \psi^2\}^{n_r} + C,$$

so that

$$\frac{dz}{dw} = \Pi A\ (w - \phi_r)^{n_r}$$

where Π is the product symbol.

It remains to find the quantities n_r. Draw a small semicircle of radius R around the point ϕ_r,



* "Ueber einige Abbildungsaufgaben" ('CRELLE,' vol. 70, 1869).

† "Sul problema delle temperature stazionarie," &c. ('Annali di Matematica,' vol. 1, 1867).

and on this semicircle let

$$w - \phi_r = \mathrm{R}(\cos\theta + i\sin\theta),$$

so that

$$\frac{dz}{dw} = \mathrm{AR}^{n_r}(\cos n_r\theta + i\sin n_r\theta).$$

Consequently, as we pass from $\theta = \pi$ to $\theta = 0$, dz/dw goes from

$$\mathrm{AR}^{n_r}(\cos n_r\pi + i\sin n_r\pi)$$

to

$$\mathrm{AR}^{n_r}.$$

The amplitude of dz/dw has, therefore, decreased by $n_r\pi$.

But the increase of amplitude is $\pi - \alpha_r$ where α_r is the internal angle of the polygon which corresponds to ϕ_r.

Therefore,

$$\pi - \alpha_r = -n_r\pi,$$

or

$$n_r = \frac{\alpha_r}{\pi} - 1,$$

so that the transformation becomes

$$\frac{dz}{dw} = \mathrm{A}\Pi(w - \phi_r)^{\alpha_r/\pi - 1},$$

which is the formula given by SCHWARZ and CHRISTOFFEL.

PROBLEM I.—SPECIAL CASE.

For the study of non-reentrant free stream lines we require a special case of this formula.

Suppose the polygon to consist of a series of straight line sinfinite in one direction, all parallel to $y = 0$, so that the angles of the polygon are either 0 or 2π, and, therefore, $n_r = +1$ or -1.

Let ϕ_{1r} correspond to an angle 2π, and, therefore, to an end of a line within a finite distance of the origin, and let ϕ_{2r} correspond to an angle 0, and, therefore, to the adjacent ends of two lines at an infinite distance from the origin.

SCHWARZ's formula then becomes

$$\frac{dz}{dw} = \Pi\mathrm{A}\frac{w - \phi_{1r}}{w - \phi_{2r}}$$

It is plain that there cannot be more than one factor more in the numerator than in the denominator, so that we can write

$$\frac{dz}{dw} = \mathrm{A}w + \mathrm{B} + \Sigma \frac{\mathrm{C}_r}{w - \phi_{2r}},$$

and, therefore, on integrating

$$z = \tfrac{1}{2}\,\mathrm{A}w^2 + \mathrm{B}w + \mathrm{D} + \Sigma\mathrm{C}_r \log (w - \phi_{2r}),$$

we may at once determine the distances between consecutive lines in terms of the ϕ_r, or in terms of the D_r.

For consider the passage of w around the small semicircle R above described.

We have

$$w - \phi_r = \mathrm{R}e^{i\theta},$$

therefore

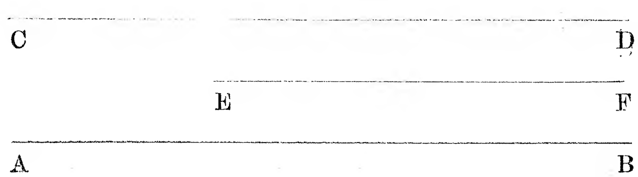
$$\begin{aligned} z_1 - z_0 &= \int_\pi^0 \frac{\mathrm{C}_r}{\mathrm{R}e^{i\theta}}\, d\,(\mathrm{R}e^{i\theta}), \\ &= i\mathrm{C}_r \int_\pi^0 d\theta, \\ &= -\,i\pi\mathrm{C}_r. \end{aligned}$$

So that y increases by $-\pi\mathrm{C}_r$ in passing the point ϕ_{2r}, and, therefore, the distance between the parallel lines r and $r+1$ is $-\pi\mathrm{C}_r$, or, in terms of the ϕ_r, is

$$-\pi\mathrm{A}\Pi\frac{\phi_r - \phi_{1s}}{\phi_r - \phi_{2s}}.$$

After we have fixed on the angle which is to correspond to $\phi = \pm\infty$, we can in general choose the position of two other points ϕ_r, ϕ_s, and then the transformation-formula is determined to an additive constant.

For example, take the case of two doubly infinite lines AB, CD, with a semi-infinite line EF between them.



We may take the zero angle (A, C) to be $\phi = \pm\infty$, and the angles (B, F) (F, D) to be $\phi = -1$, $\phi = +1$. Then if the angle E is $\phi = c$, we have

$$\frac{dz}{dw} = \frac{\mathrm{A}\,(w - c)}{(w-1)(w+1)}.$$

Let the distance between AB and EF be d_1 and that between EF and CD d_2. Then integrating past the points $\omega = -1$, $\omega = 1$, we find

$$d_1 = -\pi \frac{A(1-c)}{2},$$

$$d_2 = -\pi \frac{A(1+c)}{2},$$

which determine A and c, and, therefore, make the formula definite.

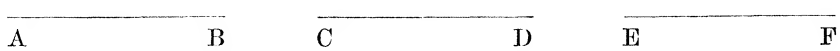
Other examples will occur in the physical application.

PROBLEM II.

The second transformation which we need may be stated most simply as an electrical problem.

Let there be any number of infinitely long plane conductors, all in the same plane, and with parallel edges.

It is required to find the potential at any point when these conductors are raised to given potentials.



Let AB, CD, EF, . . . be the sections of the conductors by the plane (xy).

Everything is symmetrical with regard to the line ABF which we take to be $y = 0$.

Consider the specifications of the transformation-function V where ψ is the potential and ϕ are the lines of force.

We must have $dV/d\psi = 0$ over the conductors, since they are straight, and, therefore, also $dV/dy = 0$.

There will be infinite points at the edges A, B . . . of the conductors, and also at points in the field corresponding to branch points of ψ (or ϕ). These last will be distributed symmetrically with respect to $y = 0$.

From these conditions it is plain that the solution is that V is the potential of masses at the singular points in question, so that we may write

$$V = \log \Pi \{(x - x_r)^2 + (y - y_r)^2\}^{n_r} + C,$$

and, therefore,

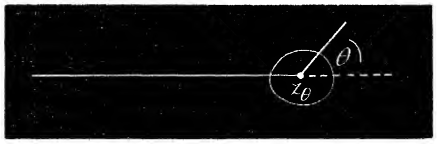
$$\frac{dz}{dw} = A\Pi (z - z_r)^{n_r}.$$

It remains to find the quantities n_r.

For a singular point in the field where m branches of ψ meet we have simply

$$n_r = -m.$$

For an edge of the conductor we may proceed thus :—



Let

$$z - z_0 = \mathrm{R}e^{i\theta},$$

then

$$(z - z_0)^n = \mathrm{R}^n (\cos n\theta + i \sin n\theta),$$

so that as θ goes from $-\pi$ to $+\pi$, $(z - z_0)^n$ goes from

$$\mathrm{R}^n [\cos(-n\pi) + i \sin(-n\pi)]$$

to

$$\mathrm{R}^n [\cos(n\pi) + i \sin(n\pi)],$$

and, therefore, the amplitude of dz/dw increases by $2n\pi$. Now the amplitude goes from 0 to π, therefore, $n = \frac{1}{2}$.

Hence

$$\frac{dw}{dz} = \Pi \frac{(z - z_r)^{m_r}}{(z - x_s)^{\frac{1}{2}}},$$

where r refers to a point in the field and s to the edge of a conductor.

If one of the conductors reduce to a line, we have two of the x_s equal, say $x_s = x_{s+1}$, and there is a factor $z - x_s$ in the denominator, and so for any number of line conductors.

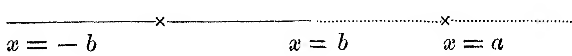
There are many other cases for which a formula like the above applies; it is not always necessary that the conductors should be in the same plane.

It is very easy to perceive such cases by considering whether the equation $d\mathrm{V}/dn = 0$ is satisfied over the conductors where dn is an element of a normal to a conductor.

By combining this transformation with that of SCHWARZ and CHRISTOFFEL we get a general solution for the free-stream-line problem, as I shall presently show.

It is necessary first to deduce some special formulæ, which will be continually used hereafter.

(a) Take first the case of one conductor and one line—



Let the conductor extend from $x = -b$ to $x = +b$ and the line distribution be at $x = a$.

Then the general formula reduces to

$$\frac{dw}{dz} = \frac{Ai}{(z-a)(z^2-b^2)^{\frac{1}{2}}}$$

where A is real, supposing that there is no singular point in the field.

The nature of the multiplier Ai is obtained by considering that between

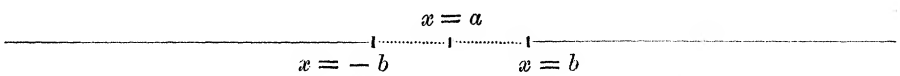
$$x = -b \text{ and } x = b \ \frac{d\psi}{dx} = 0.$$

The integral of this is

$$w = Ai \log \frac{az - b^2 + \sqrt{(a^2-b^2)}\sqrt{(z^2-b^2)}}{b(z-a)}$$

supposing the potential of the conductor is zero.

(b) Now let the conductor be two semi-infinite planes, with a gap from $x = -b$ to $x = b$ between them, and let there be a line distribution at $x = a$ where $b > a > -b$.



Then

$$\frac{dw}{dz} = \frac{A}{(z-a)(z^2-b^2)^{\frac{1}{2}}}$$

where A is real.

And

$$w = Ai \log \frac{b^2 - az + \sqrt{(b^2-a^2)}\sqrt{(b^2-z^2)}}{b(z-a)},$$

the potential of the conductor being zero.

(c) If we put $z = b + z'$ in the result of (a), and then make $b = \infty$, we get for the case of a semi-infinite conductor $x = 0$ to $x = -\infty$ with a line distribution at $x = a$,

$$w = Ai \log \frac{\sqrt{z} + \sqrt{a}}{\sqrt{z} - \sqrt{a}},$$

the potential of the conductor being 0.

These results (a), (b), (c), could of course be deduced from the known formulæ for elliptical conductors.

On the Theory of Non-reentrant Free Stream Lines.

The presence of sharp salient edges in a moving liquid always implies surfaces of discontinuity, but these may be closed or unclosed according to circumstances. It would be difficult to give a rule as to the kind of motion in a given case, for an alteration in the relative sizes of the solids concerned will totally alter the character of the motion.

As an illustration of this take the case of two parallel planes of finite breadth placed symmetrically one behind the other in a broad stream. If the second plane be of less than a certain width, free stream lines will proceed from the edges of the first, and the second will be in still water.

Now suppose the second plane broadened until it cuts the stream lines from the first plane.

The character of the motion is changed. Two vortices will appear behind the first plane, and in addition there will be free stream lines from the edge of the second plane.

No method has yet been discovered which will give solutions of cases where there is motion on both sides of a surface of discontinuity. In the problems treated in the present paper there is always still water on one side of a free stream line.

In the present section the motion considered is in two dimensions, the boundaries are plane, and the free stream lines are non-reentrant.

Let x, y be the coordinates of a point in the liquid, ϕ, ψ the potential and stream functions respectively. The region in the w plane corresponding to moving liquid in the z plane will be bounded by straight lines ψ, infinite in one direction at least and parallel to $\psi = 0$.

The area in the w plane is therefore of the nature treated in Problem I. (a), that is, it is bounded by a polygon whose angles are alternately four right angles and zero.

This area, then, by means of Problem I. (a), may be transformed into the part of a new u plane in which q is positive, where $u \equiv p + iq$.

In this u plane the boundaries of the liquid, both the plane boundaries and the free stream lines, are represented by the line $q = 0$.

Let, as before,

$$\mathrm{V} = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'} = -\log \left[\left(\frac{d\phi}{dx} \right)^2 + \left(\frac{d\phi}{dy} \right)^2 \right].$$

We have seen that V is a potential function, considered as a function of ϕ, ψ, and, therefore, it is also a potential function considered as a function of p, q, for ϕ, ψ are conjugate with respect to p, q.

Further, we have seen that along a straight boundary $\psi =$ constant we have $d\mathrm{V}/d\psi = 0$, and, since all the straight boundaries correspond to portions of $q = 0$, we

must have $dV/dq = 0$ along all these portions. Along a free stream line the pressure is constant, since it must be equal to the pressure on the liquid which is at rest.

Now, in steady motion, we have BERNOULLI'S pressure equation

$$\frac{p}{\rho} + \frac{1}{2}\left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right] = \text{constant}.$$

Therefore, along a free stream line $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is constant, and, therefore, V is constant.

All the portions of $q = 0$ which do not correspond to plane boundaries correspond to free stream lines, for which V is constant.

Lastly $\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2$ is zero at reentrant angles of the boundary, and also at points where stream lines branch.

For these points V is infinite and positive.

In all the cases we shall consider, these infinite points will be along $q = 0$, so that a stream line only branches at the boundary.

We have now reduced the problem to finding a function V which satisfies LAPLACE'S equation; is finite and continuous in that half of the plane u for which q is positive; is constant along parts of $q = 0$; along the other parts satisfies $dV/dq = 0$; and at points along it is $+\infty$.

This problem has an obvious solution.

It is plain that V is merely the potential due to conductors coinciding with those parts of $q = 0$ for which V is constant, and having that constant as potential, together with masses at the points for which V is infinite.

The general solution of this has been given in Problem II.

Let U be the conjugate of V, so that $U + iV = f(p + iq)$. Then, translating Problem II. into the present notation we have

$$\frac{d(U + iV)}{d(p + iq)} = \Pi A \frac{(u - u_r)^{m_r}}{(u - p_s)^{\frac{1}{2}}}$$

and for a point mass two factors of the denominator coincide.

Write this

$$\frac{d(U + iV)}{du} = f(u),$$

so that

$$U + iV = \int f(u)\, du.$$

Now

$$V = \log \frac{dz}{dw} \cdot \frac{dz'}{dw'},$$

therefore

$$U = i \log \left(\frac{dz}{dw} \Big/ \frac{dz'}{dw'}\right)$$

and

$$U + iV = 2i \log \frac{dz}{dw} \cdot$$

Therefore

$$\frac{dz}{dw} = e^{-i/2 \int f(u)\, du}.$$

Now we have obtained the transformation from the w to the u plane in the form

$$\frac{dw}{du} = \phi(u) \qquad \text{[Problem I. } (\alpha)\text{]}.$$

Therefore

$$\frac{dz}{du} = \frac{dz}{dw} \cdot \frac{dw}{du} = \phi(u)\, e^{-i/2 \int f(u)\, du},$$

which gives z as a function of u.

This is the general solution of the hydrodynamical problem before us.

Near an angle of the boundary or a branching of a stream line we shall have

$$dz/du = A\,(u - u_0)^n.$$

The determination of the index n rests on principles already used.

If the internal angle of the boundary, or the angle between the two branches of the stream line be α, then

$$n = \frac{\alpha}{\pi} - 1.$$

For example, if a stream line divide on a plane wall, $n = 0$, and the point of division is not a singular point for dz/du, although it is for dz/dw. We may then lay down the following rule :—

Near a singular point

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 1}$$

where α is the internal angle of the boundary except when this point is a point of branching of a stream line, in which case

$$\frac{dz}{dw} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 2},$$

for at that point

$$\frac{dz}{du} = A\,(u - u_0)^{\frac{\alpha}{\pi} - 1},$$

and

$$dw/du = B\,(u - u_0). \quad \text{[Problem I. } (\alpha)\text{]}.$$

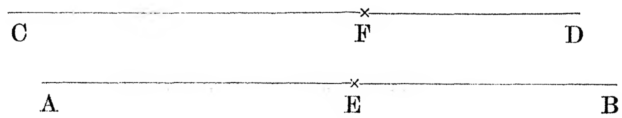
We shall now go on to consider such cases as are susceptible of tolerably simple treatment.

We shall suppose there are only two free stream lines, and throughout take the velocity along them to be 1, so that $V = 0$.

CASE I. *A Single Jet from a Vessel.*

There will be but two bounding stream lines, which we may take to be $\psi = 0$, $\psi = \pi$, both extending from $+\infty$ to $-\infty$.

The diagram in the w plane consists merely of two parallel infinite straight lines, AB, CD, at a distance π apart.



A portion of each, say EB, FD, will correspond to the boundaries of the jet.

If now we transform to the u plane so that the ends D, B go to $p = \pm\infty$, we may choose the points $u = -1$, $u = 1$, to correspond to the edges of the aperture from which the jet issues.

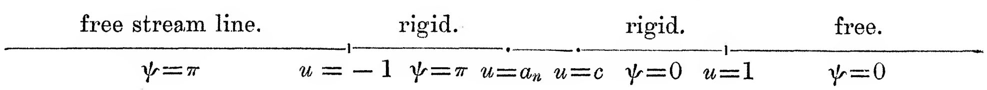
The point (C, A) will then be in a definite position, $u = c$, and the formula of transformation from w to u is

$$\frac{dw}{du} = \frac{\mathrm{A}}{u - c}.$$

Remembering that π is the distance between the two stream lines, we see that $\mathrm{A} = 1$, and therefore

$$\frac{dw}{du} = \frac{1}{u - c}.$$

Let $u = a_n$ correspond to an angle α_n of the vessel, then the points along $q = 0$ are arranged in the manner of the figure



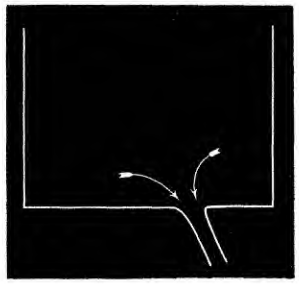
The appropriate formula is then that of Problem II. (b), viz.,

$$\frac{dz}{dw} = \Pi\left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)}\,\sqrt{(1 - u^2)}}{u - a_n}\right]^{1-\frac{\alpha_n}{\pi}}$$

and therefore

$$\frac{dz}{du} = \frac{1}{u - c}\,\Pi\left[\frac{1 - a_n u + \sqrt{(1 - a_n^2)}\,\sqrt{(1 - u^2)}}{u - a_n}\right]^{1-\frac{\alpha_n}{\pi}}$$

Example I.—A rectangular vessel of given width has an aperture in the bottom.



Here there are only two angles, each of which $= \frac{1}{2}\pi$, and therefore $1 - \alpha_n/\pi = \frac{1}{2}$. Let the two angles be $u = a$, $u = b$, where $1 > b > c > a > -1$.

Then, observing that

$$\{1 - au + \sqrt{1-a^2}\sqrt{1-u^2}\} = \{\sqrt{\tfrac{1}{2}(1-a)}\sqrt{1+u} + \sqrt{\tfrac{1}{2}(1+a)}\sqrt{1-u}\}^2$$

we get

$$\frac{dz}{du} = \frac{1}{u-c}\,\frac{\{\sqrt{\frac{1}{2}(1-a)}\sqrt{1+u} + \sqrt{\frac{1}{2}(1+a)}\sqrt{1-u}\}\,\{\sqrt{\frac{1}{2}(1-b)}\sqrt{1+u} + \sqrt{\frac{1}{2}(1+b)}\sqrt{1-u}\}}{\sqrt{(u-a\,.\,u-b)}}$$

or

$$\frac{dz}{du} = \frac{Au + B + C\sqrt{(1-u^2)}}{(u-c)\sqrt{(u-a)\,.\,(u-b)}}\,.$$

Where

$$A = \sqrt{\tfrac{1}{4}(1-a)(1-b)} - \sqrt{\tfrac{1}{4}(1+a)(1+b)},$$

$$B = \sqrt{\tfrac{1}{4}(1-a)(1-b)} + \sqrt{\tfrac{1}{4}(1+a)(1+b)},$$

$$C = \sqrt{\tfrac{1}{4}(1+a)(1-b)} + \sqrt{\tfrac{1}{4}(1-a)(1+b)}.$$

Between $u = -1$ and $u = a$, that is along the bottom of the vessel from the edge of the aperture to the angle on the right, we have

$$\frac{dy}{dp} = 0,$$

$$\frac{dx}{dp} = \frac{Ap + B + C\sqrt{(1-p^2)}}{(p-c)\sqrt{(p-a)(p-b)}}\,;$$

therefore the distance between the two points is

$$l_1 = \int_{-1}^{a} \frac{Ap + B + C\sqrt{(1-p^2)}}{(c-p)\sqrt{(p-a)(p-b)}}\,dp.$$

Similarly the other piece of the bottom of the vessel is of length

$$l_2 = \int_b^1 \frac{Ap + B + C\sqrt{(1-p^2)}}{(p-c)\sqrt{(p-a)(p-b)}}\,dp.$$

When $\phi = -\infty$, that is, when $u = c$

$$\frac{dz}{dw} = -i\,\frac{Ac + B + C\sqrt{(1-c^2)}}{\sqrt{(c-a)(b-c)}}.$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{\sqrt{(c-a)(b-c)}}{Ac + B + C\sqrt{(1-c^2)}},$$

and therefore the breadth of the vessel is

$$\pi\,\frac{Ac + B + C\sqrt{(1-c^2)}}{\sqrt{(c-a)(b-c)}} = d \text{ (say)}.$$

The breadth of the aperture is

$$d - l_1 - l_2,$$

and the breadth of the jet is ultimately π.

The question is now reduced to a matter of the integration of l_1, l_2.

For the general case elliptic integrals occur, and the expressions for our purpose may as well be left in their present form. If, however, the aperture be in the centre of the vessel, the integrals will work out, and we get a simple expression for the contraction of the jet. To this we now proceed.

Sub-Example I.—Jet from an aperture in the centre of the bottom of a rectangular vessel.

In this case we may take the angles of the vessel at $u = -a$, $u = a$, $a < 1$.

The expression for dz/du then reduces to

$$\frac{dz}{du} = \frac{1}{u}\,\frac{\sqrt{(1-a^2)} + \sqrt{(1-u^2)}}{\sqrt{u^2 - a^2}}.$$

We now have

$$l_2 = l_1 = \int_a^1 \frac{1}{p}\,\frac{\sqrt{(1-a^2)} + \sqrt{(1-p^2)}}{\sqrt{p^2-a^2}}\,dp.$$

Now

$$\int_a^1 \frac{1}{p}\,\frac{1}{\sqrt{(p^2-a^2)}}\,dp = -\int_{\frac{1}{a}}^1 \frac{d\lambda}{\sqrt{(1-a^2\lambda^2)}}\left(\lambda = \frac{1}{p}\right),$$

$$= -\frac{1}{a}\Big[\sin^{-1} a\lambda\Big]_{\frac{1}{a}}^1,$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \sin^{-1} a\right],$$

and

$$\int_a^1 \frac{\sqrt{(1-p^2)}}{p\sqrt{(p^2-a^2)}}\,dp = -\int \frac{r^2dr}{(1-r^2)\sqrt{(1-a^2-r^2)}} \quad (p^2 = 1 - r^2)$$

$$= \int \frac{dr}{\sqrt{(1-a^2-r^2)}} - \int \frac{dr}{(1-r^2)\sqrt{(1-a^2-r^2)}}$$

$$= \left[\sin^{-1}\frac{r}{\sqrt{(1-a^2)}} - \frac{1}{a}\tan^{-1}\frac{ar}{\sqrt{(1-a^2-r^2)}}\right]_{\sqrt{(1-a^2)}}^{0}$$

$$= \frac{\pi}{2}\frac{1-a}{a},$$

therefore

$$l_2 = l_1 = \frac{\pi}{2}\frac{1-a}{a} + \frac{\sqrt{(1-a^2)}}{a}\left[\frac{\pi}{2} - \sin^{-1}a\right].$$

When $u = 0$ that is, at $\phi = -\infty$, we have

$$\frac{dz}{dw} = -i\frac{\sqrt{(1-a^2)}+1}{a}.$$

Therefore the velocity in the vessel at a distance from the aperture is

$$\frac{a}{1+\sqrt{(1-a^2)}}$$

Therefore the breadth of the vessel is

$$\pi\frac{1+\sqrt{(1-a^2)}}{a} = d \text{ (say)}.$$

So that the breadth of the aperture is

$$d - 2l_1 = \pi + 2\frac{\sqrt{(1-a^2)}}{a}\sin^{-1}a.$$

Hence, since the final breadth of the jet is π, we have as the ratio of the breadth of the jet to that of the aperture

$$\frac{\pi}{\pi + 2\dfrac{\sqrt{(1-a^2)}}{a}\sin^{-1}a}$$

Now

$$\frac{1+\sqrt{(1-a^2)}}{a} = \frac{d}{\pi},$$

therefore

$$\frac{1-\sqrt{(1-a^2)}}{a} = \frac{\pi}{d},$$

and

$$2\frac{\sqrt{(1-a^2)}}{a} = \frac{d}{\pi} - \frac{\pi}{d}.$$

Also

$$\sin^{-1} a = \tan^{-1} \frac{a}{\sqrt{(1 - a^2)}} = \tan^{-1} \frac{2\pi d}{d^2 - \pi^2}.$$

Therefore the contraction ratio is

$$\frac{\pi}{\pi + \left(\frac{d}{\pi} - \frac{\pi}{d}\right) \tan^{-1} \frac{2\pi d}{d^2 - \pi^2}}.$$

Now, getting rid of the special units, let d be the breadth of the vessel as before, k the breadth of the aperture, and c that of the jet, then

$$k = c\left[1 + \frac{1}{\pi}\left(\frac{d}{c} - \frac{c}{d}\right) \tan^{-1} \frac{2dc}{d^2 - c^2}\right].$$

If d be very large compared with k, we get

$$k = c\left[1 + \frac{1}{\pi} \text{ Lim. } \frac{d}{c} \tan^{-1} 2\frac{c}{d}\right] = c\,\frac{\pi + 2}{\pi}.$$

This is the result obtained by RAYLEIGH* from KIRCHHOFF'S solution for the case of an aperture in an infinite plane bounding wall.

As d/k decreases from infinity the contraction also continually decreases, until when $d = k$ the contraction is zero.

In order to get some idea of how soon the finite breadth of the vessel affects the contraction ratio perceptibly, consider the case when

$$d^2 - c^2 = 2cd$$

or

$$d = (1 + \sqrt{2})\, c.$$

The contraction ratio is then

$$\frac{1}{1 + \frac{2}{\pi} \cdot \frac{\pi}{4}} = \tfrac{2}{3}$$

and

$$d = (1 + \sqrt{2})\, \tfrac{2}{3} k,$$

so that the finiteness of the vessel has very little effect on the jet if the breadth is more than twice that of the aperture.

The equations to the free stream line $\psi = \pi$ are

$$\left.\begin{aligned} \frac{dx}{dp} &= -\sqrt{(1 - a^2)}\,\frac{1}{p}\,\frac{1}{\sqrt{(p^2 - a^2)}} \\ \frac{dy}{dp} &= -\frac{1}{p}\,\frac{\sqrt{(p^2 - 1)}}{\sqrt{(p^2 - a^2)}} \end{aligned}\right\}$$

* 'Phil. Mag.,' Dec., 1876.

The former gives

$$x = \frac{\pi}{2} - \frac{\sqrt{(1-a^2)}}{a} \sin^{-1} \frac{a}{p}.$$

Now

$$p = -e^{\phi} = -e^{s},$$

where s is the arc. Therefore

$$x = \frac{\pi}{2} + \frac{\sqrt{(1-a^2)}}{a} \sin^{-1} ae^{-s}.$$

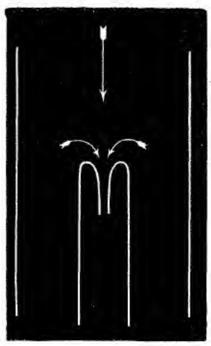
When $a = 0$ we have

$$x = \frac{\pi}{2} + e^{-s},$$

which is the form given by RAYLEIGH ('Phil. Mag.,' 1876) for KIRCHHOFF'S case.

Example II.—Tube projecting far into the bottom of a vessel of given breadth.

The tube is supposed to project so far into the vessel that the motion at its bottom may be neglected.



There are here two singular points only, $u = -a$, $u = a$, corresponding to the coalescing angles of the bottom of the vessel.

The appropriate transformation is, therefore,

$$\frac{dz}{dw} = \mathrm{A}\,\frac{1 - au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u-a}\cdot\frac{1 + au + \sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u+a}$$

$$= \mathrm{A}\,\frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}.$$

Since, when $u = \infty$, $dz/dw = -i$, we have $\mathrm{A} = i$.

So that

$$\frac{dz}{dw} = i\,\frac{2 - a^2 - u^2 + 2\sqrt{(1-a^2)}\sqrt{(1-u^2)}}{u^2 - a^2}$$

and

$$\frac{dw}{du} = \frac{1}{u} \text{ as before ;}$$

therefore

$$\frac{dz}{du} = \frac{i}{u} \frac{2 - a^2 - u^2 + 2\sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{u^2 - a^2}.$$

When $u = 0$, that is, at a distance upwards from the tube,

$$\frac{dz}{dw} = -\frac{i}{a^2}(2 - a^2 - 2\sqrt{1 - a^2}).$$

therefore the breadth of the vessel is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} = d \text{ (say)}.$$

In passing over $u = -a$, x increases by

$$2\frac{\pi}{a^2}(1 - a^2),$$

therefore the breadth of the tube is

$$\frac{\pi a^2}{2 - a^2 - 2\sqrt{(1 - a^2)}} - \frac{4\pi}{a^2}(1 - a^2) = \frac{\pi}{a^2}\{3a^2 - 2 + 2\sqrt{(1 - a^2)}\} = k \text{ (say)}.$$

Hence

$$d - k = \frac{4\pi}{a^2}(1 - a^2),$$

therefore

$$\frac{d - k + 4\pi}{4\pi} = \frac{1}{a^2}.$$

Substituting in the value of d we have

$$d + \pi = \pi\frac{d - k + 4\pi}{2\pi} + \sqrt{\left(\frac{d - k}{d - k + 4\pi}\right)} \cdot \frac{d - k + 4\pi}{2}$$

or

$$d + k - 2\pi = \sqrt{\{(d - k)(d - k + 4\pi)\}}.$$

On squaring this gives

$$dk - 2\pi d + \pi^2 = 0,$$

or, getting rid of the special unit of length, if c be the breadth of the jet, we have

$$dk - 2cd + c^2 = 0,$$

or

$$(d - c)^2 = d(d - k).$$

When d is very great we get

$$k = 2c,$$

and the ratio of contraction is $\frac{1}{2}$.

This particular case was the first solution of free stream lines given, and by HELMHOLTZ.

Lord RAYLEIGH* has given the equation

$$\frac{2}{k} = \frac{1}{c} + \frac{1}{d}$$

for the coefficient of contraction when there is a tube projecting inwards in a vessel of finite breadth.

The assumption made is, however, not that of this example, for he has taken the velocity along the bottom of the vessel to be the same as that at a distance upwards from the tube.

This is one of the very few cases in which the contraction can be determined accurately from elementary principles.

It is scarcely worth while to put the proof of this down, but it is worth remarking that the corresponding case in three dimensions, can also be worked out, viz., the case of one circular cylinder projecting far into another.

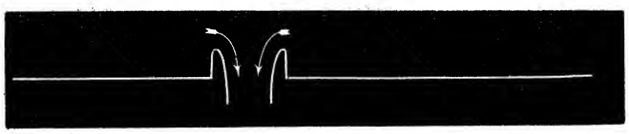
Let r_1, r_2 be the radii of the cylinders $r_1 > r_2$, and r_3 the radius of the jet, then

$$r_1^2 r_2^2 - 2r_1^2 r_3^2 + r_3^4 = 0.$$

This includes the case first noticed, I believe, by BORDA,† of a cylindrical tube in an infinite plane wall, the proof being just the same.

Example III.—Tube projecting into a vessel of great breadth.

If we take the case of a tube projecting into a vessel of great breadth, we do not get such simple expressions.



The transformation to the u plane is the same as before, but there is now a singular point at $u = 0$, corresponding to $\phi = -\infty$. We arrive at the correct result by first supposing the vessel of finite breadth, so that there are points at $u = -a$, $u = -b$, $u = b$, $u = a$, for each of which $\alpha/\pi - 1 = -\frac{1}{2}$, and then making b vanish.

We have, therefore,

$$\frac{dz}{dw} = \frac{i}{\sqrt{(u^2 - a^2)}} \{\sqrt{(1 - a^2)} + \sqrt{(1 - u^2)}\} \frac{1 + \sqrt{(1 - u^2)}}{u},$$

and

$$\frac{dz}{du} = \frac{i}{u^2\sqrt{(u^2 - a^2)}} \{\sqrt{(1 - a^2)} + 1 - u^2 + (1 + \sqrt{1 - a^2})\sqrt{(1 - u^2)}\}.$$

* "The Contracted Vein," 'Phil. Mag.,' Dec. 1876.

† 'Mém. de l'Acad.,' 1766, Paris. I owe this reference to the kindness of Lord RAYLEIGH.

Between $u = a$ and $u = 1$ we have

$$\frac{dx}{dp} = 0$$

$$\frac{dy}{dp} = \frac{1}{p^2\sqrt{(p^2 - a^2)}} \{\sqrt{(1 - a^2)} + 1 - p^2 + \sqrt{(1 - p^2)}\,(1 + \sqrt{1 - a^2})\},$$

therefore the length of the pipe is

$$(1 + \sqrt{1 - a^2}) \int_a^1 \frac{dp}{p^2\sqrt{(p^2 - a^2)}} - \int_a^1 \frac{dp}{\sqrt{(p^2 - a^2)}} + (1 + \sqrt{1 - a^2}) \int_a^1 \frac{\sqrt{(1 - p^2)}}{p^2\sqrt{(p^2 - a^2)}}\, dp.$$

Now

$$\int_a^1 \frac{dp}{p^2\sqrt{(p^2 - a^2)}} = \frac{1}{a^2}\left[\sin\theta\right]_0^{\cos^{-1}a} \qquad (p = a\sec\theta) = \frac{1}{a^2}\sqrt{(1 - a^2)}$$

$$\int_a^1 \frac{dp}{\sqrt{(p^2 - a^2)}} = \log\frac{1 + \sqrt{(1 - a^2)}}{a} = \cosh^{-1}\frac{1}{a}$$

$$\int_a^1 \frac{\sqrt{(1 - p^2)}\, dp}{p^2\sqrt{(p^2 - a^2)}} = \mathrm{L}\,(a) \text{ (say)},$$

an elliptic integral which will be brought to the standard form below.

Therefore the length of the pipe is

$$\frac{(1 + b)\, b}{a^2} - \log\frac{1 + b}{a} + (1 + b)\,\mathrm{L}\,(a),$$

where

$$b \equiv \sqrt{(1 - a^2)}.$$

From

$$u = 1 \text{ to } u = \infty$$

we have

$$\int_1^\infty \frac{dx}{dp}\, dp = (1 + b) \int_1^\infty \frac{\sqrt{(p^2 - 1)}}{p^2\sqrt{(p^2 - a^2)}}\, dp = (1 + b)\,\mathrm{M}\,(a) \text{ (say)},$$

where M is another elliptic integral to be reduced below.

Summing up, the breadth of jet is π; that of the aperture is

$$\pi + 2\,(1 + \sqrt{1 - a^2})\,\mathrm{M}\,(a),$$

so that the contraction ratio is

$$\frac{\pi}{\pi + 2\,(1 + \sqrt{1 - a^2})\,\mathrm{M}\,(a)},$$

where the length of the pipe is

$$\frac{1 + \sqrt{(1 - a^2)}}{a}\sqrt{(1 - a^2)} - \log\frac{1 + \sqrt{(1 - a^2)}}{a} + (1 + \sqrt{1 - a^2})\,\mathrm{L}\,(a).$$

To reduce the elliptic integrals, we have

$$\int_a^1 \frac{1}{p}\frac{\sqrt{(1-p^2)}}{\sqrt{(p^2-a^2)}}\,dp = \int_1^{1/a}\frac{\sqrt{(r^2-1)}}{\sqrt{(1-a^2r^2)}}\,dr \qquad rp = 1$$

$$= \int_1^{1/a}\frac{r^2-1}{\sqrt{(r^2-1\cdot 1-a^2r^2)}}.$$

Now (CAYLEY, 'Elliptic Functions,' p. 315)

$$\frac{dr}{\sqrt{(r^2-1\cdot 1-a^2r^2)}} = \frac{dx}{\sqrt{(1-x^2\cdot 1-k^2x^2)}} \text{ where } k^2 = \frac{1-a^2}{a^2} \text{ and } r^2 = \frac{1}{1-(1-a^2)x^2}.$$

Thus

$$L(a) = \int_0^1\left[\frac{1}{1-(1-a^2)x^2} - 1\right]\frac{dx}{\sqrt{(1-x^2\cdot 1-k^2x^2)}}$$

$$= \Pi_1(k, a^2-1) - F_1(k),$$

where Π and F are the third and first elliptic integrals. And

$$M(a) = \int_1^\infty \frac{\sqrt{(p^2-1)}}{p^2\sqrt{(p^2-a^2)}}\,dp = \int_0^1\frac{\sqrt{(1-r^2)}}{\sqrt{(1-a^2r^2)}}\,dr$$

$$= \left(1-\frac{1}{a^2}\right)\int_0^1\frac{dr}{\sqrt{(1-r^2\cdot 1-a^2r^2)}} + \frac{1}{a^2}\int_0^1\frac{dr\sqrt{(1-a^2r^2)}}{\sqrt{(1-r^2)}}$$

$$= \left(1-\frac{1}{a^2}\right)F_1(a) + \frac{1}{a^2}E_1(a),$$

where E is the second elliptic integral.

Suppose now the length of the pipe is small, so that a is nearly unity.

We have

$$L(a) = \int_a^1\frac{\sqrt{(1-p^2)}}{p^2\sqrt{(p^2-a^2)}}\,dp.$$

If we put $p^2 = x$, $2pdp = dx$, we may put the factor p in the integral equal to unity, and so get

$$L(a) = \tfrac{1}{2}\int_{a^2}^1\frac{\sqrt{(1-x)}}{\sqrt{(x-a^2)}}\,dx$$

$$= \tfrac{1}{4}\pi(1-a^2).$$

So that the length of pipe becomes

$$l = \left(\frac{\pi}{4}+1\right)(1-a^2) \quad \text{to the first order};$$

while the aperture becomes

$$k = \pi + 2\,(1 + \sqrt{1 - a^2})\left[\left(1 - \frac{1}{a^2}\right)\log\frac{4}{\sqrt{(1 - a)}} + 1\right]$$
$$= \pi + 2 + 2\sqrt{(1 - a^2)}$$

to the first order, and the breadth of jet is $c = \pi$.

So that

$$\frac{l}{k} = \frac{\pi + 4}{4\,(\pi + 2)}\,(1 - a^2),$$

and

$$\frac{c}{k} = \frac{\pi}{\pi + 2}\left(1 - \frac{2}{\pi + 2}\sqrt{1 - a^2}\right),$$

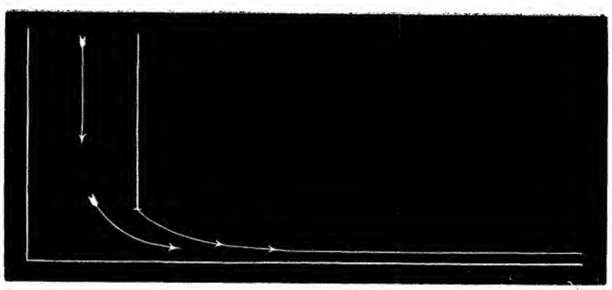
therefore

$$\frac{c}{k} = \frac{\pi}{\pi + 2}\left(1 - \frac{4}{\pi + 2}\sqrt{\frac{\pi + 2}{\pi + 4}\frac{l}{k}}\right),$$

which is the approximate expression required.

Example IV.—Flow from an aperture in a pipe in which the water is at rest.

By considering the symmetry of the motion this can be reduced to the case of flow from a rectangular vessel in which the aperture extends from the bottom up, and the bottom is continued into a horizontal plane, as in the figure.



The w diagram consists of two doubly infinite straight lines, $\psi = 0$, $\psi = \pi$, as before.

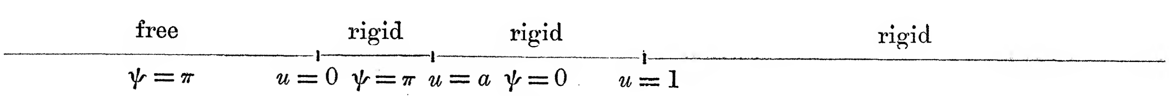
But if in the u plane we make $u = -\infty$ correspond to $\phi = \infty$, the arrangement of points is not the same as before and the notation must be altered.

We take, then, $u = 0$ for the edge of the aperture, $u = 1$ the right angle of the vessel, and then put $u = a$ for $\phi = -\infty$.

We then have

$$\frac{dw}{du} = \frac{1}{u - a}$$

and the arrangement of points on $q = 0$ is as in the figure.



The only singular point for dz/dw is at $u = 0$, and, therefore, from Problem II. (c),

$$\frac{dz}{dw} = \left(\frac{\sqrt{u}+1}{\sqrt{u}-1}\right)^{\frac{1}{2}}$$

$$\frac{dz}{du} = \frac{1}{u-a}\left(\frac{\sqrt{u}+1}{\sqrt{u}-1}\right)^{\frac{1}{2}}$$

Along the free stream line

$$\frac{dz}{dp} = \frac{1}{p-a}\left(\frac{p+1+2i\sqrt{(-p)}}{p-1}\right)^{\frac{1}{2}}$$

where p lies between 0 and $-\infty$.

If

$$\frac{dy}{dx} = \tan\psi$$

we have

$$\frac{p+1}{p-1} = \cos 2\psi$$

$$\frac{2\sqrt{(-p}}{p-1} = \sin 2\psi$$

$$p = -\cot^2\psi.$$

Therefore

$$\frac{dx}{d\psi} = 2\frac{\cos^2\psi\ \mathrm{cosec}^3\psi}{\cot^2\psi + a}$$

$$\frac{dy}{d\psi} = 2\frac{\cos\psi\ \mathrm{cosec}^2\psi}{\cot^2\psi + a}.$$

If we put $\cos\psi = \lambda$ in the first we get

$$\begin{aligned} x &= -2\int\frac{\lambda^2}{1-\lambda^2}\,\frac{1}{a+(1-a)\lambda^2}\,d\lambda, \\ &= -2\int\frac{d\lambda}{1-\lambda^2} - 2\,\frac{a}{1-2a}\int\frac{d\lambda}{a+(1-a)\lambda^2}, \\ &= 2\log\frac{1-\lambda}{1+\lambda} - 2\,\frac{\sqrt{a}}{1-2a}\cdot\frac{1}{\sqrt{(1-a)}}\tan^{-1}\frac{\lambda}{\sqrt{a}}\sqrt{(1-a)} + \mathrm{C}, \end{aligned}$$

and, in the second, putting

$$\sin\psi = \mu$$

we have

$$\begin{aligned} y &= 2\int\frac{d\mu}{1-(1-a)\mu^2}, \\ &= -\frac{1}{\sqrt{(1-a)}}\log\frac{1-\sqrt{(1-a)}\,\mu}{1+\sqrt{(1-a)}\,\mu} + \pi. \end{aligned}$$

Hence when $\psi = \frac{1}{2}\pi$, *i.e.*, at the edge of the aperture

$$y = \frac{1}{\sqrt{(1-a)}} \log \frac{1+\sqrt{(1-a)}}{1-\sqrt{(1-a)}} + \pi,$$

therefore the ratio of the jet to the aperture is

$$\frac{c}{k} = \frac{\pi}{\pi + \dfrac{1}{\sqrt{(1-a)}} \log \dfrac{1+\sqrt{(1-a)}}{1-\sqrt{(1-a)}}}.$$

The velocity at $\phi = -\infty$ is

$$\left(\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)^{\frac{1}{2}},$$

therefore the width of the vessel is

$$d = \pi \left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{\frac{1}{2}},$$

therefore

$$\frac{d}{c} = \left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{\frac{1}{2}}.$$

From which we obtain

$$\frac{1+\sqrt{a}}{2} = \frac{1}{\dfrac{c^2}{d^2}+1},$$

$$\frac{1-\sqrt{a}}{2} = \frac{1}{\dfrac{d^2}{c^2}+1},$$

therefore

$$k = c\left[1 + \frac{1}{2\pi}\left(\frac{c}{d}+\frac{d}{c}\right) \log \frac{\frac{1}{2}\left(\dfrac{c}{d}+\dfrac{d}{c}\right)+1}{\frac{1}{2}\left(\dfrac{c}{d}+\dfrac{d}{c}\right)-1}\right],$$

or

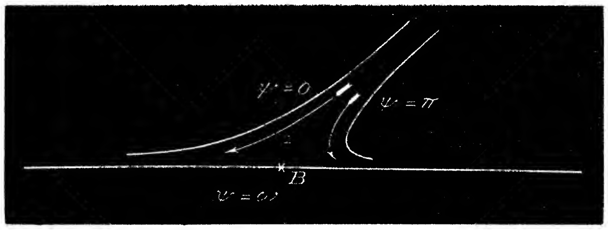
$$k = c\left[1 + \frac{1}{\pi}\left(\frac{d}{c}+\frac{c}{d}\right) \log \frac{d+c}{d-c}\right].$$

This is for the vessel considered. For the pipe

$$k = c\left[1 + \frac{1}{\pi}\left(\frac{2d}{c}+\frac{c}{2d}\right) \log \frac{2d+c}{2d-c}\right].$$

CASE II.—*Impact of a stream against a plane.*

A stream of given breadth impinges at a given angle against a plane.



The impinging stream is bounded by the stream lines $\psi = 0$, $\psi = \pi$.

The stream line which branches at the point B on the plane is $\psi = \varpi$.

The diagram in the w plane consists of two infinite straight lines with a semi-infinite one between them, as in the figure.

$\psi = \pi$

$\psi = \varpi$

$\psi = 0.$

In transforming to the u plane we suppose that $\phi = -\infty$ corresponds to $u = \pm\infty$, and that $u = -1$, $u = 1$ are the extremities of the plane.

We must then take $u = a$, an unknown constant for the point B, where the stream line $\psi = \varpi$ divides, and observe that $a < 1 > -1$.

We then have

$$\frac{dw}{du} = \mathrm{A}\,\frac{u - a}{(u - 1)(u + 1)}$$

where

$$\left.\begin{aligned} -\pi\mathrm{A}\,\frac{1 + a}{2} &= \varpi \\ -\pi\mathrm{A}\,\frac{1 - a}{2} &= \pi - \varpi \end{aligned}\right\},$$

therefore

$$\mathrm{A} = -1 \quad \text{and} \quad a = \frac{2\varpi}{\pi} - 1.$$

Along $q = 0$ we have, therefore, the following arrangement of points :—

free		rigid		rigid		free	
$\phi = -\infty$ $\psi = 0$	$u = -1$	$\psi = \varpi$	$u = a$		$u = 1$	$\psi = \pi$	$\phi = -\infty.$

The only singular point of dz/dw is at $u = a$.

Hence

$$\frac{dz}{dw} = \frac{1 - au + \sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{u - a},$$

and, therefore,

$$\frac{dz}{du} = -\frac{1 - au + \sqrt{(1 - a^2)}\sqrt{(1 - u^2)}}{(u - 1)(u + 1)}$$

$$= \frac{1 - au}{1 - u^2} + \sqrt{(1 - a^2)}\frac{1}{\sqrt{(1 - u^2)}}.$$

When $u = \pm \infty$, we have

$$\frac{dx}{d\phi} + i\frac{dy}{d\phi} = -a - i\sqrt{(1 - a^2)}.$$

Therefore, if θ be the inclination of the stream to the plane, we have

$$\tan\theta = \frac{\sqrt{(1 - a^2)}}{a} \quad \text{or} \quad \cos\theta = a.$$

Since we have before found

$$a = \frac{2\varpi}{\pi} - 1,$$

this gives

$$\pi\cos\theta = 2\varpi - \pi.$$

This equation merely expresses that the momentum parallel to the plane is unaltered by impact.

Along the stream line $\psi = \pi$ we have

$$\left.\begin{aligned} \frac{dx}{dp} &= -\frac{1 - ap}{p^2 - 1} \\ \frac{dy}{dp} &= \sqrt{(1 - a^2)}\frac{1}{\sqrt{(p^2 - 1)}}, \end{aligned}\right\}$$

therefore

$$x = \frac{a}{2}\log(p^2 - 1) + \tfrac{1}{2}\log\frac{p + 1}{p - 1} + \mathrm{A}$$

$$\left.\begin{aligned} &= \frac{1 + a}{2}\log(p + 1) - \frac{1 - a}{2}\log(p - 1) + \mathrm{A} \\ y &= \sqrt{(1 - a^2)}\log(p + \sqrt{p^2 - 1}) + \pi - \varpi. \end{aligned}\right\}$$

where p lies between $+1$ and ∞; and for $\psi = 0$ we have

$$\frac{dx}{dp} = -\frac{1 - ap}{p^2 - 1} \qquad \frac{dy}{dp} = -\sqrt{(1 - a^2)}\frac{1}{\sqrt{(p^2 - 1)}},$$

therefore

$$\left.\begin{aligned} x &= \frac{1+a}{2}\log -(1+p) - \frac{1-a}{2}\log(1-p) + \mathrm{A}' \\ y &= \sqrt{(1-a^2)}\log(\sqrt{p^2-1}-p) + \varpi \end{aligned}\right\}$$

where p lies between -1 and $-\infty$.

If we put $p = \dfrac{1}{\cos\theta}$ in the first and $p = -\dfrac{1}{\cos\theta}$ in the second, we get for $\psi = \pi$

$$\left.\begin{aligned} x_1 &= (1+a)\log\cos\frac{\theta}{2} - (1-a)\log\sin\frac{\theta}{2} - a\log\cos\theta + \mathrm{A} \\ y_1 &= \sqrt{(1-a^2)}\log\cot\tfrac{1}{2}\left(\frac{\pi}{2}-\theta\right) + \pi - \varpi, \end{aligned}\right\}$$

and for $\psi = 0$

$$\left.\begin{aligned} x_2 &= (1+a)\log\sin\frac{\theta}{2} - (1-a)\log\cos\frac{\theta}{2} - a\log\cos\theta + \mathrm{A}' \\ y_2 &= \sqrt{(1-a^2)}\log\cot\tfrac{1}{2}\left(\frac{\pi}{2}-\theta\right) + \varpi \end{aligned}\right\}$$

where in both cases θ lies between 0 and $\frac{1}{2}\pi$. When $\theta = \frac{1}{2}\pi$ in both we get

$$\left.\begin{aligned} x_1 - x_2 &= \mathrm{A} - \mathrm{A}' \\ y_1 - y_2 &= \pi - 2\varpi \end{aligned}\right\}$$

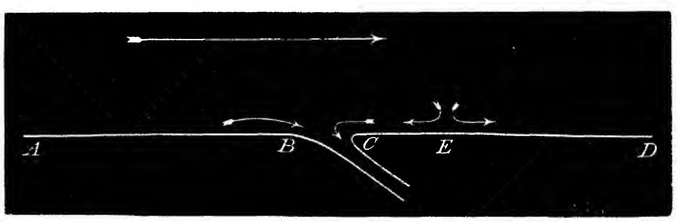
therefore

$$\mathrm{A} - \mathrm{A}' = \frac{\pi + (\pi - 2\varpi)\,a}{\sqrt{(1-a^2)}} = \pi\sqrt{(1-a^2)},$$

and the equations of the bounding stream lines are now completely determined.

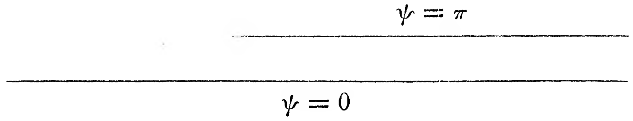
Case III.—*Flow of a broad stream past a plane wall in which there is an aperture.*

Let BC be the aperture in a plane wall ABCD, and let the stream flow from left to right.



The left boundary of the issuing jet will be the continuation of the stream line AB. The right boundary will be one branch of a stream line which divides on the plane CD at some point E.

The diagram in the w plane will then consist of one infinite straight line and one semi-infinite, as in the figure.



We suppose $\psi = 0$, $\psi = \pi$ to be the bounding stream lines. In transforming to the u plane we take the point $\psi = 0$, $\phi = -\infty$ to be $u = -\infty$; $\psi = 0$, $\phi = \infty$ to be $u = a$, and then use the two arbitrary constants at our disposal by making the edges of the aperture to be $u = -b$, $u = b$, and the branch point of $\psi = \pi$ to be $u = 1 + a$, where $1 + a > b > a$.

We then have

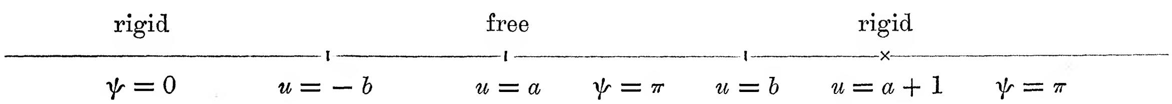
$$\frac{dw}{du} = \mathrm{A}\,\frac{u - a - 1}{u - a},$$

and, remembering that the bounding stream lines are $\psi = 0$, $\psi = \pi$, we obtain $\mathrm{A} = 1$, so that

$$\frac{dw}{du} = \frac{u - a - 1}{u - a}.$$

There is only one singular point for dz/dw, viz., $u = a + 1$, and at this point $\alpha/\pi - 1 = 0$, so that no factor $u - a - 1$ appears in dz/du.

The arrangement of points on $q = 0$ is as in the figure



and hence

$$\frac{dz}{dw} = \frac{(a + 1)\,u - b^2 + \sqrt{(a + 1)^2 - b^2}\,\sqrt{(u^2 - b^2)}}{b\,(u - a - 1)},$$

so that

$$\frac{dz}{du} = \frac{(a + 1)\,u - b^2 + \sqrt{(a + 1)^2 - b^2}\,\sqrt{(u^2 - b^2)}}{b\,(u - a)}.$$

When $u = \pm\infty$, we have

$$\frac{dz}{dw} = \frac{a + 1 + \sqrt{(a + 1)^2 - b^2}}{b},$$

and, therefore, the velocity of the stream is

$$\frac{b}{(a + 1) + \sqrt{(a + 1)^2 - b^2}}.$$

When $u = a$, that is at a great distance along the jet,

$$\frac{dz}{dw} = -\frac{a(a+1) - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}}{b},$$

so that the inclination of the jet to the wall is ultimately

$$\cos^{-1}\frac{b^2 - a(a+1)}{b},$$

the final breadth of the jet being π.

There appear to be two constants here, whereas there ought only to be one.

The explanation is that we have not yet expressed that the two parts of the boundary are in the same plane. This will give a relation between a and b.

Between $u = -b$ and $u = a$ we have

$$\frac{dx}{dp} = \frac{1}{b}\frac{(a+1)p - b^2}{p - a},$$

$$\frac{dy}{dp} = \frac{\sqrt{(a+1)^2 - b^2}}{b}\quad\frac{\sqrt{(b^2 - p^2)}}{p - a}.$$

Now

$$\int\frac{\sqrt{(b^2 - p^2)}}{p - a} = \int\frac{1}{\sqrt{(b^2 - p^2)}}\left[\frac{b^2 - a^2}{p - a} - a - p\right],$$

and

$$\begin{aligned}
&\int\frac{dp}{(p - a)\sqrt{(b^2 - p^2)}}\\
&= -\int\frac{dr}{\sqrt{(b^2 - a^2)r^2 + 2ar - 1}}, \qquad \text{where } p - a = -\frac{1}{r},\\
&= -\int\frac{dr}{\sqrt{\left(\sqrt{b^2 - a^2}\,r + \frac{a}{\sqrt{(b^2 - a^2)}}\right)^2 - \frac{b^2}{b^2 - a^2}}},\\
&= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\left[\sqrt{(b^2 - a^2)}\,r + \frac{a}{\sqrt{(b^2 - a^2)}} + \sqrt{(b^2 - a^2)r^2 + 2ar - 1}\right],\\
&= -\frac{1}{\sqrt{(b^2 - a^2)}}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{a - p}.
\end{aligned}$$

Thus along $\psi = 0$, measuring from the edge A, we have

$$x - x_{\text{A}} = \frac{a+1}{b}(p + b) + \frac{a(a+1) - b^2}{b}\log\frac{a - p}{a + b},$$

$$y - y_{\text{A}} = \frac{\sqrt{(a + 1^2) - b^2}}{b}\left[-\sqrt{(b^2 - a^2)}\log\frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(a - p)b}\right.$$
$$\left. + a\sin^{-1}.\frac{-p}{b} - a\frac{\pi}{2} + \sqrt{(b^2 - p^2)}\right].$$

In like manner between $u = a$ and $u = b$ on $\psi = \pi$, we have

$$x - x_B = \frac{a+1}{b}(p - b) + \frac{a(a+1) - b^2}{b} \log \frac{p-a}{b-a},$$

$$y - y_B = \frac{\sqrt{(a+1)^2 - b^2}}{b}\left[-\sqrt{(b^2 - a^2)} \log \frac{b^2 - ap + \sqrt{(b^2 - a^2)(b^2 - p^2)}}{(p-a)b} \right.$$
$$\left. - a \sin^{-1}\frac{p}{b} + a\frac{\pi}{2} + \sqrt{(b^2 - p^2)}\right].$$

To find $x_A - x_B$ and $y_A - y_B$ we must put $p = a - \epsilon$ and $p = a + \epsilon$ respectively in the above, and then decrease ϵ indefinitely.

If we then remember that, in passing the point $p = a$, $x + iy$ increases by

$$-\frac{i\pi}{b}\left\{(a+1)a - b^2 + i\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}\right\}$$

we get

$$x_B - x_A = 2(a+1) + \frac{a(a+1) - b^2}{b} \log \frac{b-a}{b+a} + \frac{\pi}{b}\sqrt{(a+1)^2 - b^2}\sqrt{(b^2 - a^2)}$$

and

$$y_B - y_A = -\pi a \frac{\sqrt{(a+1)^2 - b^2}}{b} + \frac{\pi}{b}\{b^2 - a(a+1)\}.$$

In the case we are considering $y_B = y_A$, and, therefore,

$$b^2 - a(a+1) = a\sqrt{(a+1)^2 - b^2},$$

or

$$b^2 = (a+1)^2 - 1,$$

so that

$$\left.\begin{array}{l}(a+1)^2 - b^2 = 1 \\ a(a+1) - b^2 = -a.\end{array}\right\}$$

Using these equalities, we now get for the breadth of the aperture

$$x_B - x_A = 2(a+1) + \frac{a}{b}\log\frac{b+a}{b-a} + \pi\frac{\sqrt{2a}}{b}$$

where $b/(a+2)$ is the velocity of the stream, π the breadth of the jet, and $\cos^{-1}(a/b)$ the inclination of the jet to the plane.

To get rid of the special units let v_1 be the velocity of the stream, v_2 of the jet, k the breadth of the aperture, c that of the jet.

Then

$$\frac{v_1}{v_2} = \frac{b}{a+2},$$

therefore

$$\left(\frac{v_1}{v_2}\right)^2 = \frac{b^2}{(a+2)^2} = \frac{a}{a+2}.$$

So that

$$a = \frac{2v_1^{\,2}}{v_2^{\,2} - v_1^{\,2}},$$

and

$$b = \frac{2v_2 v_1}{v_2^{\,2} - v_1^{\,2}},$$

therefore

$$a + b = \frac{2v_1(v_2 + v_1)}{v_2^{\,2} - v_1^{\,2}}$$

$$-a + b = \frac{2v_1(v_2 - v_1)}{v_2^{\,2} - v_1^{\,2}}.$$

Hence

$$\frac{k}{c}\pi = 2\frac{v_2^{\,2} + v_1^{\,2}}{v_2^{\,2} - v_1^{\,2}} + \frac{v_1}{v_2}\log\frac{v_2 + v_1}{v_2 - v_1} + \pi\sqrt{\frac{v_2^{\,2} - v_1^{\,2}}{v_2^{\,2}}},$$

and the inclination of the jet to the plane is

$$\cos^{-1}\frac{v_1}{v_2}.$$

An interesting element not yet calculated is the distance of the branching of the stream line $\psi = \pi$ from the edge of the aperture.

This distance is

$$\int_b^{a+1}\left[\frac{1}{b}\frac{(a+1)p - b^2 + \sqrt{(p^2 - b^2)}}{p - a}\right]dp$$

$$= \int_b^{a+1}\left[\frac{a+1}{b} - \frac{a}{b}\frac{1}{p-a} + \frac{1}{b}\left\{p + a - \frac{2a}{p-a}\right\}\frac{1}{\sqrt{(p^2 - b^2)}}\right]dp,$$

where we have used the relation between a and b to simplify the expression,

$$= \left[\frac{a+1}{b}p - \frac{a}{b}\log(p-a) + \frac{1}{b}\sqrt{(p^2 - b^2)} + \frac{a}{b}\log\left(p + \sqrt{p^2 - b^2}\right) + \frac{\sqrt{2a}}{b}\sin^{-1}\left\{\frac{b^2 - ap}{b(p-a)}\right\}\right]_b^{a+1}$$

$$= \frac{a+1}{b}(a + 1 - b) + \frac{a}{b}\log(b - a) + \frac{1}{b} + \frac{a}{b}\log\frac{a+2}{b} + \frac{\sqrt{2a}}{b}\left(\sin^{-1}\frac{a}{b} - \frac{\pi}{2}\right).$$

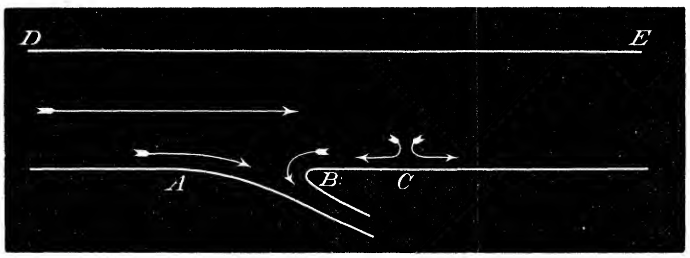
Therefore, in terms of v_1/v_2 and c, this distance is l where

$$\frac{l}{c}\pi = \left[\frac{v_2^2 + v_1 v_2 + v_1^2}{v_1 v_2} \times \frac{v_2 - v_1}{v_2 + v_1} + \frac{v_1}{v_2} \log \frac{2v_2}{v_2 + v_1} + \sqrt{\frac{v_2^2 - v_1^2}{v_2^2}}\left(\sin^{-1}\frac{v_1}{v_2} - \frac{\pi}{2}\right)\right].$$

For example, let the stream have half the velocity of the issuing jet, so that $v_1 = \frac{1}{2} v_2$. Then the jet makes an angle of 60° with the plane, and its breadth is $\frac{1}{2} k$ approximately, while l is about $\frac{1}{4} k$.

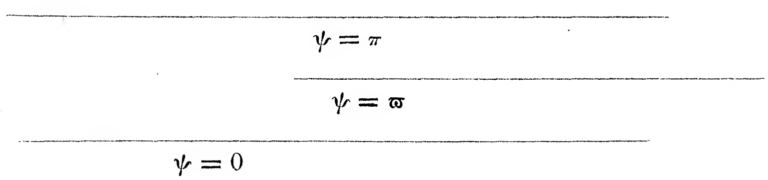
CASE IV.—*Jet from a pipe along which liquid is flowing.*

The liquid is flowing along a pipe bounded by the plane walls AC, DE, and there is an aperture AB in the former.



The left boundary of the jet is $\psi = 0$, the right boundary is a stream line $\psi = \varpi$, which branches at a point C on BC, and DE is $\psi = \pi$.

The w diagram is as in the figure, consisting of two infinite lines, $\psi = 0$, $\psi = \pi$, with a semi-infinite line $\psi = \varpi$ between them.



In transforming to the u plane, we suppose that $\phi = -\infty$ corresponds to $u = -\infty$, and that $u = -1$, $u = +1$ are the edges of the aperture.

The constants of the transformation are then determined, and we take $u = a$ for the branch point, $u = b$ for the jet at an infinite distance, $u = c$ for $\phi = \infty$ in the pipe.

Then

$$\frac{dw}{du} = \text{A} \frac{u - a}{(u - b)(u - c)} \qquad b < a < c,$$

where

$$-\text{A} \frac{b - a}{b - c} \pi = \varpi$$

$$-\text{A} \frac{c - a}{b - c} \pi = \varpi - \pi,$$

and therefore

$$A = -1$$

$$\frac{\varpi}{\pi} = \frac{b-a}{b-c}$$

$$\frac{\pi - \varpi}{\pi} = -\frac{c-a}{b-c}.$$

The arrangement of points on $q = 0$ is as in the figure

rigid		free		free		rigid		rigid
$\psi=0$	$u=-1$	$\psi=0$	$u=b$	$\psi=\varpi$	$u=1$	$u=a$	$u=c$	$\psi=\pi$

and from Problem II. (α) we have

$$\frac{dz}{dw} = \frac{au - 1 + \surd(a^2-1)\ \surd(u^2-1)}{u-a},$$

so that

$$\frac{dz}{du} = -\frac{au - 1 + \surd(a^2-1)\ \surd(u^2-1)}{(u-b)\ (u-c)}.$$

Hence, in the pipe, the velocity at $\phi = -\infty$ is

$$\frac{1}{a + \surd(a-1)} = a - \surd(a^2-1),$$

and at $\phi = \infty$ is

$$\frac{c-a}{ac - 1 + \surd(a^2-1)\surd(c^2-1)}.$$

The breadth of the pipe is, therefore,

$$\pi\left(a + \sqrt{a^2-1}\right) = (\pi - \varpi)\,\frac{ac - 1 + \surd(a^2-1)\ \surd(c^2-1)}{c-a}$$

Now

$$\pi - \varpi = \pi\,\frac{c-a}{c-b},$$

therefore

$$(c-b)\left(a + \sqrt{a^2-1}\right) = ac - 1 + \surd(a^2-1)\ \surd(c^2-1).$$

This is the relation between a, b, and c. To find the size of the aperture we must integrate between $u = -1$ and $u = 1$. Within these limits we have

$$\frac{dx}{dp} = \frac{1-ap}{(b-p)\ (c-p)} = \frac{1-ab}{c-b}\,\frac{1}{b-p} + \frac{ac-1}{c-b}\,\frac{1}{c-p}$$

$$\frac{dy}{dp} = -\surd(a^2-1)\,\frac{\surd(1-p^2)}{(b-p)\ (c-p)} = -\frac{\surd(a^2-1)}{c-b}\,\sqrt{1-p^2}\left[\frac{1-ab}{b-p} + \frac{ac-1}{c-p}\right].$$

The former gives

$$x - x_A = -\frac{1-ab}{c-b}\log\frac{b-p}{b+1} - \frac{ac-1}{c-b}\log\frac{c-p}{c+1} \quad \text{between } p = -1 \text{ and } p = b,$$

and

$$x - x_B = \frac{ab-1}{c-b}\log\frac{p-b}{1-b} - \frac{ac-1}{c-b}\log\frac{c-p}{c-1} \quad \text{between } p = b \text{ and } p = 1.$$

Also in passing the point $u = b$, x changes by

$$\pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b};$$

therefore, putting $p = b - \epsilon$ in $x - x_A$, $p = b + \epsilon$ in $x - x_B$, and proceeding to the limit, we get as the breadth of the aperture,

$$\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{c+1}{c-1} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}.$$

The direction of the jet makes an angle

$$\cos^{-1}\left[\frac{1-ab}{a-b}\right]$$

with the bounding planes. The breadth of the jet is

$$\pi\frac{a-b}{c-b}.$$

To sum up: Let d be the breadth of the pipe, k of the aperture, l of the jet. Let, further, v_1 be the velocity at $\phi = -\infty$, v_2 that at $\phi = \infty$, v_3 that of the jet.

$$\frac{v_3}{v_1} = a + \sqrt{(a^2-1)} \quad \dots \dots \quad (1)$$

$$\frac{v_1}{v_2} = \frac{c-b}{c-a} \quad \dots \dots \quad (2)$$

and therefore

$$\frac{v_1}{v_1 - v_2} = \frac{c-b}{a-b} \quad \dots \dots \quad (3)$$

$$\frac{d}{l} = \frac{v_3}{v_1 - v_2} \quad \dots \dots \quad (4)$$

$$\frac{k}{l} = \frac{c-b}{\pi(a-b)}\left\{\frac{1-ab}{c-b}\log\frac{1+b}{1-b} + \frac{ac-1}{c-b}\log\frac{1+c}{-1+c} + \pi\frac{\sqrt{(a^2-1)}\sqrt{(1-b^2)}}{c-b}\right\} \quad . \quad (5)$$

with

$$(c-b)(a+\sqrt{a^2-1}) = ac - 1 + \sqrt{(a^2-1)}\sqrt{(c^2-1)} \quad \dots \quad (6)$$

The equation (6) may be written

$$(c-a)\frac{v_3}{v_2} = (ac-1) + \sqrt{(a^2-1)}\sqrt{(c^2-1)}.$$

Now

$$(c-a)^2 \equiv (ac-1)^2 - (a^2-1)(c^2-1),$$

therefore

$$(c-a)\frac{v_2}{v_3} = (ac-1) - \sqrt{(a^2-1)}\sqrt{(c^2-1)};$$

and, by addition and subtraction,

$$(c-a)\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right) = 2(ac-1),$$

$$(c-a)\left(\frac{v_3}{v_2}-\frac{v_2}{v_3}\right) = 2\sqrt{(a^2-1)}\sqrt{(c^2-1)}.$$

From (1)

$$a = \tfrac{1}{2}\left(\frac{v_3}{v_1}+\frac{v_1}{v_3}\right)$$

$$\sqrt{(a^2-1)} = \tfrac{1}{2}\left(\frac{v_3}{v_1}-\frac{v_1}{v_3}\right).$$

Therefore

$$c\left[\frac{v_3}{v_2}+\frac{v_2}{v_3}-\frac{v_3}{v_1}-\frac{v_1}{v_3}\right] = \tfrac{1}{2}\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right)\left(\frac{v_3}{v_1}+\frac{v_1}{v_3}\right)-2,$$

or,

$$c(v_1-v_2)(v_3{}^2-v_1v_2) = \tfrac{1}{2}(v_1-v_2)^2 v_3 + \tfrac{1}{2}(v_3{}^2-v_1v_2)^2\frac{1}{v_3};$$

so that

$$\frac{1+c}{-1+c} = \frac{[(v_1-v_2)v_3+(v_3{}^2-v_1v_2)]^2}{[-(v_1-v_2)v_3+(v_3{}^2-v_1v_2)]^2}$$

$$= \frac{(v_3-v_2)^2(v_3+v_1)^2}{(v_3+v_2)^2(v_3-v_1)^2}.$$

And

$$\frac{ac-1}{a-b} = \tfrac{1}{2}\left(\frac{v_3}{v_2}+\frac{v_2}{v_3}\right)\frac{v_2}{v_1-v_2}.$$

Also

$$(c-a)\frac{v_1}{v_2} = c-b \quad \text{from (2)},$$

therefore

$$b = \tfrac{1}{2}\frac{v_3{}^2+v_1{}^2}{v_2v_3} - \tfrac{1}{2}\frac{(v_1-v_2)^2v_3{}^2+(v_3{}^2-v_1v_2)^2}{v_2v_3(v_3{}^2-v_1v_2)}$$

$$= \tfrac{1}{2}\frac{(v_1{}^2+v_1v_2)(v_3{}^2-v_1v_2)-(v_1-v_2)^2v_3{}^2}{v_2v_3(v_3{}^2-v_1v_2)},$$

so that

$$1 + b = \tfrac{1}{2} \frac{(v_1^2 + v_1 v_2 + 2v_2 v_3)(v_3^2 - v_1 v_2) - (v_1 - v_2)^2 v_3^2}{v_2 v_3 (v_3^2 - v_1 v_2)},$$

and

$$1 - b = \tfrac{1}{2} \frac{(2v_2 v_3 - v_1^2 - v_1 v_2)(v_3^2 - v_1 v_2) + (v_1 - v_2)^2 v_3^2}{v_2 v_3 (v_3^2 - v_1 v_2)}.$$

These may be reduced to the form

$$1 + b = \tfrac{1}{2} \frac{(v_3 + v_1)^2}{v_3^2 - v_1 v_2} \cdot \frac{2v_3 - v_1 - v_2}{v_3},$$

$$1 - b = \tfrac{1}{2} \frac{(v_3 - v_1)^2}{v_3^2 - v_1 v_2} \cdot \frac{2v_3 + v_1 + v_2}{v_3}.$$

Further,

$$a - b = \tfrac{1}{2} \frac{(v_3^2 v_2 - v_1^3)(v_3^2 - v_1 v_2) + (v_1 - v_2)^2 v_3^2 v_1}{v_1 v_2 v_3 (v_3^2 - v_1 v_2)},$$

$$= \tfrac{1}{2} \frac{(v_3^2 - v_1^2)^2}{v_1 v_3 (v_3^2 - v_1 v_2)},$$

$$1 - ab = \tfrac{1}{4} \frac{\{4v_3^2 v_1 v_2 - (v_3^2 + v_1^2)(v_1^2 + v_1 v_2)\}\{v_3^2 - v_1 v_2\} + (v_3^2 + v_1^2)(v_1 - v_2)^2 v_3^2}{v_1 v_2 v_3^2 (v_3^2 - v_1 v_2)},$$

$$= \tfrac{1}{4} \frac{(v_1 + v_2)(v_3^2 - v_1^2)^2}{v_1 v_3^2 (v_3^2 - v_1 v_2)}.$$

Whence, finally,

$$\frac{1 - ab}{a - b} = \tfrac{1}{2} \frac{v_1 + v_2}{v_3},$$

$$\frac{1 + b}{1 - b} = \frac{(v_3 + v_1)^2}{(v_3 - v_1)^2} \frac{2v_3 - v_1 - v_2}{2v_3 + v_1 + v_2},$$

$$\frac{\sqrt{(a^2 - 1)}\sqrt{(1 - b^2)}}{c - b} = \tfrac{1}{2} \frac{\sqrt{4v_3^2 - (v_1 + v_2)^2}}{v_3}.$$

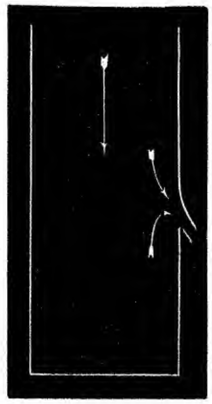
So that we get for k/l

$$\frac{k}{l}\pi = \tfrac{1}{2} \frac{v_1 + v_2}{v_3} \log \frac{(v_3 + v_1)^2}{(v_3 - v_1)^2} \frac{2v_3 - v_1 - v_2}{2v_3 + v_1 + v_2}$$

$$+ \frac{v_3^2 + v_2^2}{v_3 (v_1 - v_2)} \log \frac{(v_3 - v_2)(v_3 + v_1)}{(v_3 + v_2)(v_3 - v_1)}$$

$$+ \tfrac{1}{2}\pi \left\{\frac{4v_3^2 - (v_1 + v_2)^2}{v_3^2}\right\}^{\frac{1}{2}}.$$

An interesting special case is the flow from an aperture in the side of a rectangular vessel, of which the bottom is at a considerable distance from the aperture.



For this case $v_2 = 0$, and, therefore,

$$\frac{d}{l} = \frac{v_3}{v_1},$$

expressing the equality of inflow and outflow.

So that

$$\frac{k}{l}\pi = \frac{1}{2}\frac{v_1}{v_3}\log\left(\frac{v_3+v_1}{v_3-v_1}\right)^2 \cdot \frac{2v_3-v_1}{2v_3+v_1} + \frac{v_3}{v_1}\log\frac{v_3+v_1}{v_3-v_1} + \frac{\pi}{2}\left\{4 - \frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{v_1}{v_3} + \frac{v_3}{v_1}\right)\log\frac{v_3+v_1}{v_3-v_1} + \frac{1}{2}\frac{v_1}{v_3}\log\frac{2v_3-v_1}{2v_3+v_1} + \frac{\pi}{2}\left\{4 - \frac{v_1^2}{v_3^2}\right\}^{\frac{1}{2}},$$

or

$$\frac{k}{l}\pi = \left(\frac{d}{l} + \frac{l}{d}\right)\log\frac{d+l}{d-l} + \frac{1}{2}\frac{l}{d}\log\frac{2d-l}{2d+l} + \frac{\pi}{2}\left(4 - \frac{l^2}{d^2}\right)^{\frac{1}{2}},$$

which is the simplest expression for this case.

The angle the jet makes with the bounding wall is

$$\cos^{-1}\frac{1-ab}{a-b},$$

or

$$\cos^{-1}\frac{1}{2}\frac{v_1+v_2}{v_3}$$

in the general case, while in the particular case it is simply

$$\cos^{-1}\frac{1}{2}\frac{v_1}{v_3}.$$

PART II.

We will now go on to consider problems in which the region of (x, y) is not simply connected, and consequently SCHWARZ's transformation does not apply.

First consider the area outside a closed polygon.

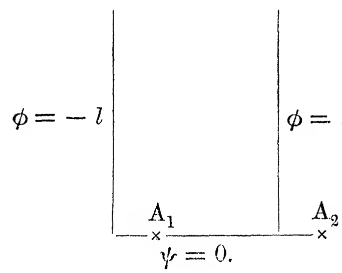
We may state the problem electrically thus :—

PROBLEM III.

To find the potential due to a polygonal prismatic conductor at a given potential which we may take to be zero, the field at infinity being at an infinite potential.

Let ψ be the potential, ϕ the lines of force, and let ϕ increase by $2l$ in going round the polygon.

Then the area in the ω plane is a rectangle bounded by $\psi = 0$, $\psi = \infty$, $\phi = -l$, $\phi = l$.



The conditions which the transformation function V satisfy are

(a) $$\frac{dV}{d\psi} = 0 \text{ over } \psi = 0.$$

(b) V finite and continuous at all points within a finite distance in the rectangle.

(c) V periodic in ϕ so that
$$V(\phi + 2l) = V(\phi).$$

(d) V infinite at points $A_1 \ldots$ along $\psi = 0$.

We can determine V from these specifications by means of W. THOMSON's method of images. For if we repeat the points $A_1 \ldots$ at equal distances $2l$ along $\psi = 0$, and make V the potential of these points, the conditions will clearly be satisfied.

Hence
$$V = A\Sigma_r \log \prod_{-\infty}^{+\infty} \{(\phi - \phi_r - 2nl)^2 + \psi^2\}^m,$$

and
$$\frac{dz}{dw} = A\Pi_r \prod_{n\,-\infty}^{\;\;+\infty} \{w - \phi_r - 2nl\}^m$$
$$= B\Pi_r \left\{\sin(w - \phi_r)\frac{\pi}{2l}\right\}^m,$$

and exactly as in the treatment of SCHWARZ's formula we have $m = \alpha_r/\pi - 1$, where α_r is the exterior angle of the polygon corresponding to ϕ_r.

So that, finally,*

$$\frac{dz}{dw} = \mathrm{B}\Pi\left\{\sin(w - \phi_r)\frac{\pi}{2l}\right\}^{\alpha_r/\pi - 1}$$

As an example, take the case of a rectangle.

The four singular points may be taken to be

$$\phi = -a, \qquad \phi = a, \qquad \phi = l - a, \qquad \phi = l + a,$$

and we have, taking for simplicity $l = \frac{1}{2}\pi$,

$$\frac{dz}{dw} = \mathrm{A}\sqrt{-\sin(w - a)\sin(w + a)\cos(w - a)\cos(w + a)},$$

$$= \tfrac{1}{2}\mathrm{A}\sqrt{(\sin^2 2a - \sin^2 2w)}.$$

Hence

$$z = \tfrac{1}{2}\mathrm{A}\int\sqrt{(\sin^2 2a - \sin^2 2w)}\,dw + \mathrm{B},$$

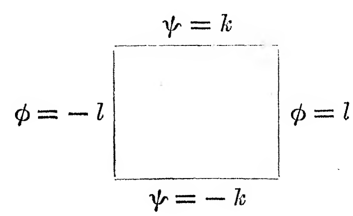
and z is an elliptic integral of w.

PROBLEM IV.

Suppose now there are two polygonal prismatic conductors, one at potential $\psi = -k$, the other at $\psi = k$, and at first (A) suppose that one of the conductors is within the other.

Let ϕ increase by $2l$ in going round either polygon.

The area in the w plane is now a finite rectangle bounded by $\phi = -l$, $\phi = l$, $\psi = -k$, $\psi = k$.



In order to satisfy the condition $d\mathrm{V}/d\psi = 0$ over $\psi = -k$ and $\psi = k$ we must have a double system of images of the singular points, viz., at

$$\phi_0 + 2ml, \qquad \psi_0 + 4nk,$$
$$\phi_0 + 2ml, \qquad \psi_0 + (2n + 1)\,2k.$$

* Mr. BRILL has already given this formula ('Messenger of Math.,' August, 1889), and I only insert it here for the sake of completeness.

NOTE.—April 29. Since the above paper was read, Mr. BRILL has given the next transformation in the same journal.

Thus each singular point gives a factor

$$\prod_{-\infty}^{+\infty}\prod_{-\infty}^{+\infty}\{w - w_0 - 2ml - 4nk\}^{\mathrm{M}}\{w - w_0 - 2ml - (2n + 1)\,2k\}^{\mathrm{M}},$$

or, what is the same thing,

$$\Theta\,[a\,(w - w_0)]^{\mathrm{M}}\,\mathrm{H}\,[a\,(w - w_0)]^{\mathrm{M}},$$

where Θ, H are JACOBI'S functions so indicated, and, therefore,

$$\frac{dz}{dw} = \Pi\,\{\Theta\,[a\,(w - w_0)]\,\mathrm{H}\,[a\,(w - w_0)]\}^{\mathrm{M}},$$

where

$$\left.\begin{aligned} al &= \mathrm{K} \\ 2ak &= \mathrm{K}' \end{aligned}\right\}$$

K, K′ being the complete elliptic integrals usually so denoted.

If α_0 be the internal angle of the figure corresponding to w_0, we have, as before,

$$\mathrm{M} = \frac{\alpha_0}{\pi} - 1,$$

so that, finally,

$$\frac{dz}{dw} = \Pi_r\,\{\Theta\,[a\,(w - w_r)]\,\mathrm{H}\,[a\,(w - w_r)]\}^{\alpha_r/\pi - 1}.$$

(B) Suppose now that one conductor is outside the other, and that the potential at infinity is zero, that of the conductors being $-k$ and $+k$. We suppose equal and opposite quantities of electricity on the conductors, $2l$ being the cyclic constant, as before.

The terms corresponding to angles of the polygon will be the same as in (A).

But there is now in addition a singular point in the field which we proceed to determine.

At a great distance from the prisms the potential will be the same as for two line distributions at the centres of mass, say at $z = a$, $z = -a$.

So that

$$w = \mathrm{M}\log\frac{z - a}{z + a} = -2\,\frac{\mathrm{M}a}{z}$$

ultimately, and

$$\frac{dw}{dz} = 2\mathrm{M}az^{-2} = \frac{w^2}{2\mathrm{M}a}$$

or

$$\frac{dz}{dw} = \frac{2\mathrm{M}a}{w^2},$$

and, therefore, there is a point of order -2 at the point in the w rectangle corresponding to the potential at infinity.

Let this point be ϕ_0, ψ_0. Then we have

$$\frac{dz}{dw} = \frac{\Pi_r \{\Theta\left[a(w - w_r)\right] \mathrm{H}\left[a(w - w_r)\right]\}^{\alpha_r/\pi - 1}}{\{\Theta\left[a(w - w_0)\right] \mathrm{H}\left[a(w - w_0)\right]\}^2}$$

as the general expression for the potential of two polygonal prismatic conductors exterior to each other.

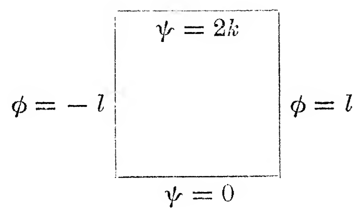
Hollow Vortices.

In conclusion I shall show how the methods of this paper may be applied to find the form of hollow vortices.

Inside a vessel bounded by plane walls, let there be a hollow vortex in steady motion.

Let $\psi = 0$ be the free steam line of the vortex, $\psi = 2k$ the rigid boundary.

In the w plane the area is bounded by $\psi = 0$, $\psi = 2k$, $\phi = -l$, $\phi = l$, $2l$ being the circulation round the vortex.



The function V satisfies the following conditions :—

(a.) $\mathrm{V} = 0$ over $\psi = 0$, if the velocity along the free steam line be unity.

(b.) $d\mathrm{V}/d\psi = 0$ over $\psi = 2k$.

(c.) V is periodic with respect to ϕ, so that $\mathrm{V}(\phi + 2l) = \mathrm{V}(\phi)$.

These conditions are to be satisfied by taking equal singular points at distances $2l$ along $\psi = 2k$, and then continually reflecting these points in the two planes $\psi = 0$, $\psi = 2k$, but in reflecting in $\psi = 0$ the image is of opposite sign to the object.

Corresponding, then, to a point M at $(\phi_0, 2k)$ we have positive images at

$$\phi_0 + 2ml, \quad 2k + 2m \cdot 4k,$$

and negative images at

$$\phi_0 + 2ml, \quad -2k + 2m \cdot 4k.$$

Therefore, corresponding to this point M, we have a factor

$$\frac{\mathrm{H}^{\mathrm{M}} a(w - \phi_0 - 2ik)}{\Theta^{\mathrm{M}} a(w - \phi_0 - 2ik)} \quad \text{in} \quad \frac{dz}{dw},$$

where

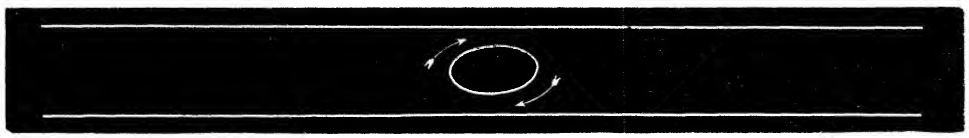
$$\left.\begin{aligned} al &= \mathrm{K} \\ 4ak &= \mathrm{K}' \end{aligned}\right\}$$

nd

$$M = \frac{\alpha_0}{\pi} - 1.$$

It is quite clear that all but the simplest cases will be of quite unmanageable omplexity.

One of the very simplest cases may be taken as an example.



Example.—Hollow vortex between two parallel planes.

Here for the two singular points $\alpha = 0$.

If for simplicity we put $a = 1$, and, therefore, $4k = K'$, the singular points are at $= 0$, $\phi = K$.

Hence

$$\frac{dz}{dw} = C\,\frac{\Theta\left(w - \frac{iK'}{2}\right)\,\Theta\left(w - \frac{iK'}{2} - K\right)}{H\left(w - \frac{iK'}{2}\right)\,H\left(w - \frac{iK'}{2} - K\right)}$$

$$= C'\,\frac{1}{sn\left(w - \frac{iK'}{2}\right)\,sn\left(w - \frac{iK'}{2} - K\right)}$$

$$= C''\,\frac{dn\left(w - \frac{iK'}{2}\right)}{sn\left(w - \frac{iK'}{2}\right)\,cn\left(w - \frac{iK'}{2}\right)}.$$

This integrates at once, giving

$$z = C'' \log \frac{sn\,(w - \frac{1}{2}iK')}{cn\,(w - \frac{1}{2}iK')} + C'''$$

$$\equiv C'' \log \mathrm{tn}\,(w - \tfrac{1}{2}iK') + C'''$$

The equation to the vortex is, therefore,

$$z = C'' \log \mathrm{tn}\,(\phi - \tfrac{1}{2}iK') + C'''.$$